Math 360 – Linear Algebra **Fall 2008**
9-10-08 Class Notes

Matrices

As we have already seen, a ***matrix*** is a rectangular array of numbers. If a matrix A has m columns and n rows, we say that its ***dimensions*** are $m \times n$ (or that it is an $\boldsymbol{m \times n}$ ***matrix***). An arbitrary $m \times n$ matrix would be represented as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

or simply,

$$A = \left[a_{ij} \right].$$

Of course, if the matrix is named M or B (or whatever), then the entries names change too. Notice that an n-dimensional vector is simply an $n \times 1$ matrix.

Definition: Two $m \times n$ matrices A and B are ***equal*** if every entry is the same (i.e. if $a_{ij} = b_{ij}$ for all i and j).

Matrices are a very fundamental tool in mathematics. To be able to effectively utilize them, we need to be adept at matrix arithmetic. We have three operations we can perform using matrices: scalar multiplication, matrix addition, and matrix multiplication. Scalar multiplication and matrix addition work exactly the same as they did for vectors (not surprisingly). Note that to add two matrices, they must have the same dimensions. Matrix multiplication is a different story.

To multiply two matrices together, we're need to recall the dot product of two vectors. The dot product $\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.

Note that the dot product of two vectors is a scalar (indeed it is sometimes called the ***scalar product***). Also note that in order to be able to get the dot product, the number of columns in the first vector must equal the number of rows in the second vector. This is true when we multiply arbitrary matrices as well. Given two matrices A and B, in order for them to be "multiply-able" the number of columns in A must equal the number of rows in B. So let's say that A is an $m \times n$ matrix and B is an $n \times k$ matrix. Notice that those "interior" dimensions (the columns of A and the rows of B) match. To compute the product AB, we form the $m \times k$ matrix obtained by computing the dot product of each row of A with each column of B. Let's do an example.

Example 1: Multiply $\begin{bmatrix} 2 & 5 & 1 & -2 \\ 0 & 6 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 6 & 1 & 2 \\ 0 & 8 & 4 \\ 5 & 6 & 1 \end{bmatrix}$.

I picked a couple of nice, big matrices so we would be able to see what's going on. First of all note that we CAN multiply these together because the first matrix is 2×4 and the second is 4×3. (This seems like a good time to mention the apparent fact that matrix multiplication is NOT commutative. Notice that the dimensions would not work for us to switch the order of these two matrices.)

The product of these two matrices will be a 2×3 matrix.

$$\begin{bmatrix} 2 & 5 & 1 & -2 \\ 0 & 6 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 6 & 1 & 2 \\ 0 & 8 & 4 \\ 5 & 6 & 1 \end{bmatrix} = \begin{bmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \end{bmatrix}$$

To fill in each blank, we need to compute the dot product of the corresponding row and column of the given matrices. For example, to fill in the blank in the 1^{st} row, 1^{st} column, we take the dot product of the 1^{st} row and 1^{st} column of the two matrices.

$$2 \cdot 1 + 5 \cdot 6 + 1 \cdot 0 + (-2) \cdot 5 = 22,$$

so we have,

$$\begin{bmatrix} 2 & 5 & 1 & -2 \\ 0 & 6 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 6 & 1 & 2 \\ 0 & 8 & 4 \\ 5 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 22 & \_ & \_ \\ \_ & \_ & \_ \end{bmatrix}.$$

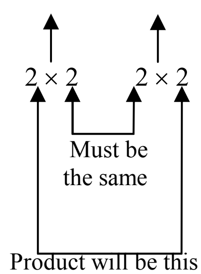
Let's pick another entry to compute. How do we calculate the entry in the 1st row, 3rd column? That's right. You find the dot product of the 1st row and the 3rd column of the two given matrices.

$$2 \cdot (-1) + 5 \cdot 2 + 1 \cdot 4 + (-2) \cdot 1 = 10.$$

Continuing in this fashion, we get the product,

$$\begin{bmatrix} 2 & 5 & 1 & -2 \\ 0 & 6 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 6 & 1 & 2 \\ 0 & 8 & 4 \\ 5 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 22 & 1 & 10 \\ 51 & 0 & 3 \end{bmatrix}.$$

Example 2: Multiply $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$.



Since the interior dimensions match (both are 2), we can multiply these together. The product will also be a 2×2 matrix. By taking the dot product of each row of the first matrix with each column of the second, we get the product

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 21 & 13 \end{bmatrix}.$$

It is interesting to note that with these two particular matrices, you COULD reverse the order and still be able to multiply them. In that case, we would get,

$$\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 14 & 22 \\ 5 & 8 \end{bmatrix}.$$

Clearly, as mentioned above, ***matrix multiplication is not commutative***.

Example 3: Write the matrix equation $\begin{bmatrix} 3 & 1 & -2 \\ 2 & 2 & 1 \\ 8 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$ as a system of equations.

If we multiply out the left-hand side we get $\begin{bmatrix} 3x_1 + x_2 - 2x_3 \\ 2x_1 + 2x_2 + x_3 \\ 8x_1 \quad + 3x_3 \end{bmatrix}$. Since matrices are equal when all entries are equal, the matrix equation turns into the system of equations,

$$\begin{aligned} 3x_1 + x_2 - 2x_3 &= 5 \\ 2x_1 + 2x_2 + x_3 &= 4 \\ 8x_1 \qquad + 3x_3 &= 3 \end{aligned}$$

Definition: The $\boldsymbol{n \times n}$ ***identity matrix*** $\boldsymbol{I}$ is the matrix with 1's down the main diagonal and 0's elsewhere. So $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}$. An $n \times n$ matrix A is ***invertible*** (or ***nonsingular***) if there exists a matrix B such that $AB = BA = I$. In this case, B is called the ***(multiplicative) inverse*** of A. If there is no such matrix, A is called ***singular***.

Theorem 1: Let α, β be scalars and let $A, B,$ and C be matrices. Then each of the following statements are true whenever the operations are defined (i.e. when the dimensions match).

(a) $A+B=B+A$ (matrix addition is commutative)
(b) $A+(B+C)=(A+B)+C$ (matrix addition is associative)
(c) $A(BC)=(AB)C$ (matrix multiplication is associative)
(d) $A(B+C)=AB+AC$ and $(A+B)C=AC+BC$ (distributivity)
(e) $(\alpha\beta)A=\alpha(\beta A)$
(f) $\alpha(AB)=(\alpha A)B=A(\alpha B)$
(g) $(\alpha+\beta)A=\alpha A+\beta A$ and $\alpha(A+B)=\alpha A+\alpha B$ (distributivity)

Definition: Let $A=\left[a_{ij}\right]$ be an $m\times n$ matrix. The ***transpose of A*** is the $n\times m$ matrix $A^T=\left[x_{ij}\right]$ where $x_{ij}=a_{ji}$. (In other words, to form the transpose, you interchange rows and columns). A matrix A is ***symmetric*** if $A=A^T$.

Example 4: Find the transpose of each matrix.

(a) $A=\begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix}$ (b) $B=\begin{bmatrix} 3 & 2 & -1 \\ 0 & 6 & 5 \\ 4 & -5 & 2 \end{bmatrix}$

(a) $A^T=\begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$ (so A is not symmetric)

(b) $B=\begin{bmatrix} 3 & 0 & 4 \\ 2 & 6 & -5 \\ -1 & 5 & 2 \end{bmatrix}$ (so B is symmetric)

Theorem 2: There four basic rules involving transposes.

(a) $(A^T)^T=A$
(b) $(\alpha A)^T=\alpha A^T$
(c) $(A+B)^T=A^T+B^T$
(d) $(AB)^T=B^T A^T$

Homework

1. For each problem compute (if possible) $A+3B$, $B-A$, and AB. If something is impossible to compute, explain why.

(a) $A=\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 2 \\ -12 & 3 \end{bmatrix}$ (b) $A=\begin{bmatrix} -5 & 0 & 2 \\ 1 & 1 & 7 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix}$

(c) $A=\begin{bmatrix} 0 & 3 \\ 7 & -1 \\ -2 & -3 \\ 4 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 6 & 1 \\ 8 & 2 \end{bmatrix}$

2. Let $A=\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Compute A^2 and A^3. What do you think A^n will be?

3. Find two 2×2 matrices A and B such that $AB=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

4. Find nonzero matrices A, B, and C for which $AC=BC$ and $A\neq B$.

5. Explain why each of the following arithmetic rules will not work in general when the real numbers a and b are replaced by $n\times n$ matrices A and B.

(a) $(a+b)^2=a^2+2ab+b^2$ (b) $(a+b)(a-b)=a^2-b^2$

6. The matrix $M=\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ has the property that $M^2=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Is it possible for a nonzero symmetric B matrix to have this property? Explain your answer.

7. A matrix A is ***skew-symmetric*** if $A^T=-A$. Show that if a matrix is skew-symmetric then all its diagonal entries must be 0.